YET ANOTHER HOPF INVARIANT

JEAN-PAUL DOERAENE AND MOHAMMED EL HAOUARI

Abstract. The classical Hopf invariant is defined for a map $f\colon S^r \to X$. Here we define 'hcat' which is some kind of Hopf invariant built with a construction in Ganea's style, valid for maps not only on spheres but more generally on a 'relative suspension' $f\colon \Sigma_A W \to X$. We study the relation between this invariant and the sectional category and the relative category of a map. In particular, for $\iota_X : A \to X$ being the 'restriction' of f on A, we have $\mathrm{relcat}\,\iota_X \leqslant \mathrm{hcat} f \leqslant \mathrm{relcat}\,\iota_X + 1$ and $\mathrm{relcat}\, f \leqslant \mathrm{hcat} f$.

Our aim here is to make clearer the link between the Lusternik-Schnirelmann category (cat), more generally the 'relative category' (relcat), closely related to James'sectional category (secat), and the Hopf invariants. In order to do this, we introduce a new integer, namely hcat, that combines the Iwaze's version of Hopf invariant [3], based on the *difference up to homotopy between two maps* defined for a given section of a Ganea fibration, and the framework of the sectional and relative categories, searching for the *least integer* such that the Ganea fibration has a section, possibly with additional conditions. To do this combination, we simply define our invariant hcat, as the least integer such that the Ganea fibration has a section σ with additional condition that the corresponding two maps ($f \circ \sigma$ and ω_n in this paper) are homotopic.

It appears that for $f\colon S^r \to X$ or even for $f\colon \Sigma W \to X$, we obtain an integer that can be either $\mathrm{cat}\,(X)$, or $\mathrm{cat}\,(X) + 1$. More generally, for any $f\colon \Sigma_A W \to X$, we have $\mathrm{relcat}\,(f \circ \theta) \leqslant \mathrm{hcat}(f) \leqslant \mathrm{relcat}\,(f \circ \theta) + 1$, where $\theta\colon A \to \Sigma_A W$ is the map arising in the construction of $\Sigma_A W$.

In section 2, we study the influence of hcat in a homotopy pushout. In section 3, we introduce the 'strong' version of our invariant, and we obtain another important inequality: for any $f\colon \Sigma_A W \to X$, we have $\mathrm{relcat}\,(f) \leqslant \mathrm{hcat}(f)$. In section 4, we give alternative equivalent conditions to get hcat. Applications and examples are given.

1. The Hopf category

We work in the category of pointed topological spaces. All constructions are made up to homotopy. A 'homotopy commutative diagram' has to be understood in the sense of [4].

Recall the following construction:

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Definition 1. For any map $\iota_X\colon A \to X$, the *Ganea construction* of ι_X is the following sequence of homotopy commutative diagrams ($i \geqslant 0$):

$$\begin{tikzcd}
 & A \arrow[dr, "\alpha_{i+1}"] \arrow[rr, bend left, "\iota_X"] & & X \\
F_i \arrow[ur, "\eta_i"] \arrow[dr, "\beta_i"'] & & G_{i+1} \arrow[r, "g_{i+1}"] & X \\
 & G_i \arrow[ur, "\gamma_i"] \arrow[urr, bend right, "g_i"'] & &
\end{tikzcd}$$

where the outside square is a homotopy pullback, the inside square is a homotopy pushout and the map $g_{i+1} = (g_i, \iota_X)\colon G_{i+1} \to X$ is the whisker map induced by this homotopy pushout. The iteration starts with $g_0 = \iota_X\colon A \to X$. We set $\alpha_0 = \mathrm{id}_A$.

For any $i \geqslant 0$, there is a whisker map $\theta_i = (\mathrm{id}_A, \alpha_i)\colon A \to F_i$ induced by the homotopy pullback. Thus we have the sequence of maps $A \xrightarrow{\theta_i} F_i \xrightarrow{\eta_i} A$ and θ_i is a homotopy section of η_i. Moreover we have $\gamma_i \circ \alpha_i \simeq \alpha_{i+1}$, thus also $\alpha_{i+1} \simeq \gamma_i \circ \gamma_{i-1} \circ \cdots \circ \gamma_0$.

We denote by $\gamma_{i,j}\colon G_i \to G_j$ the composite $\gamma_{j-1} \circ \cdots \circ \gamma_{i+1} \circ \gamma_i$ (for $i < j$) and set $\gamma_{i,i} = \mathrm{id}_{G_i}$.

Of course, everything in the Ganea construction depends on ι_X. We sometimes denote G_i by $G_i(\iota_X)$ to avoid ambiguity.

Definition 2. Let $\iota_X\colon A \to X$ be any map.

1) The *sectional category* of ι_X is the least integer n such that the map $g_n\colon G_n(\iota_X) \to X$ has a homotopy section, i.e. there exists a map $\sigma\colon X \to G_n(\iota_X)$ such that $g_n \circ \sigma \simeq \mathrm{id}_X$.

2) The *relative category* of ι_X is the least integer n such that the map $g_n\colon G_n(\iota_X) \to X$ has a homotopy section σ and $\sigma \circ \iota_X \simeq \alpha_n$.

3) The *relative category of order* k of ι_X is the least integer n such that the map $g_n\colon G_n(\iota_X) \to X$ has a homotopy section σ and $\sigma \circ g_k \simeq \gamma_{k,n}$.

We denote the sectional category by $\mathrm{secat}\,(\iota_X)$, the relative category by $\mathrm{relcat}\,(\iota_X)$, and the relative category of order k by $\mathrm{relcat}_k\,(\iota_X)$. If $A = *$, $\mathrm{secat}\,(\iota_X) = \mathrm{relcat}\,(\iota_X)$ and is denoted simply by $\mathrm{cat}\,(X)$; this is the 'normalized' version of the Lusternik-Schnirelmann category.

Clearly, $\mathrm{secat}\,(\iota_X) \leqslant \mathrm{relcat}\,(\iota_X)$. We have also $\mathrm{relcat}\,(\iota_X) \leqslant \mathrm{relcat}_1\,(\iota_X)$, see Proposition 6 below.

In the sequel, we will consider a given homotopy pushout:

$$\begin{tikzcd}
W \arrow[r, "\eta"] \arrow[d, "\beta"'] & A \arrow[d, "\theta"] \\
A \arrow[r, "\theta"'] & \Sigma_A W
\end{tikzcd}$$

In other words, the map θ is a map such that $\mathrm{Pushcat}\,\theta \leq 1$ in the sense of [2]. We call this homotopy pushout a 'relative suspension' because in some sense, A plays the role of the point in the ordinary suspension.

We also consider any map $f\colon \Sigma_A W \to X$, and set $\iota_X = f \circ \theta$.

We don't assume $\eta \simeq \beta$ in general. This is true, however, if θ is a homotopy monomorphism, and in this case we can 'think' of ι_X as the 'restriction' of f on A.

For $n \geqslant 1$, consider the following homotopy commutative diagram:

$$(\dagger)\qquad \begin{tikzcd}
 & W \arrow[rr, "\beta"] \arrow[dl, "\eta"'] \arrow[dd, dashed] & & A \arrow[dl, "\theta"'] \arrow[dr, "\theta"] \arrow[dd, "\alpha_{n-1}"] & \\
A \arrow[rr, "\theta"] \arrow[dd, equal] & & \Sigma_A W \arrow[rr, equal] \arrow[dd, dotted, "\omega_n"] & & \Sigma_A W \arrow[dd, "f"] \\
 & F_{n-1}(\iota_X) \arrow[rr] \arrow[dl] & & G_{n-1}(\iota_X) \arrow[dl, "\gamma_{n-1}"] \arrow[dr, "g_{n-1}"] & \\
A \arrow[rr, "\alpha_n"'] & & G_n(\iota_X) \arrow[rr, "g_n"'] & & X
\end{tikzcd}$$

where the map $W \to F_{n-1}$ is induced by the bottom outer homotopy pullback and the map $\omega_n\colon \Sigma_A W \to G_n$ is induced by the top inner homotopy pushout. We have $f \simeq g_n \circ \omega_n$ by the 'Whiskers maps inside a cube' lemma (see [2], Lemma 49). Also notice that $\alpha_n \simeq \omega_n \circ \theta \simeq \gamma_{n-1} \circ \alpha_{n-1}$; so $\omega_n \simeq (\alpha_n, \alpha_n)$ is the whisker map of two copies of α_n induced by the homotopy pushout $\Sigma_A W$. Finally, for all $k \geqslant 1$, we can see that $\omega_n \simeq \gamma_{k,n} \circ \omega_k$.

Definition 3. The *Hopf category* of f is the least integer $n \geqslant 1$ such that $g_n\colon G_n(\iota_X) \to X$ has a homotopy section $\sigma\colon X \to G_n(\iota_X)$ such that $\sigma \circ f \simeq \omega_n$.

We denote this integer by $\operatorname{hcat}(f)$.

Actually, speaking of 'Hopf category of f' is a misuse of language. We should speak of 'Hopf category of the datas η, β and f'.

Example 4. Let $X = \Sigma_A W$ and $f \simeq \mathrm{id}_X$. Then, as might be expected, $\operatorname{hcat}(f) = 1$. Indeed, in this case, as $g_1 \circ \omega_1 \simeq f \simeq \mathrm{id}_X$, ω_1 is a homotopy section of g_1. Moreover, $\omega_1 \circ f \simeq \omega_1 \circ \mathrm{id}_X \simeq \omega_1$, so $\operatorname{hcat}(f) = 1$.

Example 5. Let $X \not\simeq *$ and $W = A \vee A$, $\beta \simeq \mathrm{pr}_1 : A \vee A \to A$ and $\eta \simeq \mathrm{pr}_2 : A \vee A \to A$ the obvious maps. Then $\Sigma_A W \simeq *$ and we have no choice for f that must be the null map $f\colon * \to X$. In this case the condition $\sigma \circ f \simeq \omega_n$ is always satisfied, so $\operatorname{hcat}(f) = \operatorname{secat}(\iota_X) = \operatorname{cat}(X)$.

Notice that relcat is a particular case of hcat: When $W = A$, $\eta \simeq \beta \simeq \mathrm{id}_A$, then $\iota_X \simeq f$, $\omega_n \simeq \alpha_n$ and $\operatorname{hcat}(f) = \operatorname{relcat}(\iota_X)$. Also relcat_1 is a particular case of hcat: When $W = F_0$, then $\Sigma_A W \simeq G_1$, $\theta \simeq \gamma_0 \simeq \alpha_1$, and if, moreover, $f \simeq g_1$, then $\omega_n \simeq \gamma_{1,n}$ and $\operatorname{hcat}(f) = \operatorname{relcat}_1(\iota_X)$.

The following proposition shows that these particular cases are in fact lower and upper bounds for $\operatorname{hcat}(f)$.

Proposition 6. *Whatever can be f (and $\iota_X = f \circ \theta$), we have*

$$\operatorname{secat}(f) \leqslant \operatorname{relcat}(\iota_X) \leqslant \operatorname{hcat}(f) \leqslant \operatorname{relcat}_1(\iota_X) \leqslant \operatorname{relcat}(\iota_X) + 1.$$

Proof. Consider the following homotopy commutative diagram ($n \geqslant 1$):

$$\begin{tikzcd}
 & & G_n \arrow[dd, bend left, "g_n"] \\
A \arrow[r, "\theta"] \arrow[urr, bend left, "\alpha_n"] \arrow[drr, bend right, "\iota_X"'] & \Sigma_A W \arrow[ur, "\omega_n"'] \arrow[dr, "f"] & \\
 & & X
\end{tikzcd}$$

We see that if there is a map $\sigma\colon X \to G_n$ such that $\omega_n \simeq \sigma \circ f$ then $\alpha_n \simeq \sigma \circ \iota_X$ and this proves the second inequality.

Now consider the following homotopy commutative diagram ($n \geqslant 1$):

$$\begin{tikzcd}
 & & G_n \arrow[dd, bend left, "g_n"] \\
\Sigma_A W \arrow[r, "\omega_1"] \arrow[urr, bend left, "\omega_n"] \arrow[drr, bend right, "f"'] & G_1 \arrow[ur, "\gamma_{1,n}"] \arrow[dr, "g_1"'] & \\
 & & X
\end{tikzcd}$$

We see that if there is a map $\sigma\colon X \to G_n$ such that $\gamma_{1,n} \simeq \sigma \circ g_1$ then $\omega_n \simeq \sigma \circ f$ and this proves the third inequality.

The first inequality comes from $\operatorname{secat}(f) \leqslant \operatorname{secat}(\iota_X) \leqslant \operatorname{relcat}(\iota_X)$, the first of these two inequalities comes from [2], Proposition 29.

Finally, the fourth inequality is proved in [1]. □

So $\operatorname{hcat}(f)$ establishes a 'dichotomy' between maps $f\colon \Sigma_A W \to X$:

- Either $\operatorname{hcat}(f) = \operatorname{relcat}(\iota_X)$ and we have a σ such that $f \circ \sigma \simeq \omega_n$ already for $n = \operatorname{secat}(\iota_X)$;
- either $\operatorname{hcat}(f) = \operatorname{relcat}(\iota_X) + 1$ and we have a σ such that $f \circ \sigma \simeq \omega_n$ only for $n > \operatorname{secat}(\iota_X)$

Our last example of the section shows that the inequalities of Proposition 6 can be strict, and even that two may be strict at the same time:

Example 7. Let $X = *$, $A \not\simeq *$ and consider $\iota_*\colon A \to *$. We have $G_i(\iota_*) \simeq A \bowtie \ldots \bowtie A$, the join of $i+1$ copies of A. For any k, $\gamma_{k,k} \simeq \mathrm{id}$, so it cannot factorize through $*$; but $\gamma_{k,k+1}$ is homotopic to the null map, so $\operatorname{relcat}_k(\iota_*) = k+1$. Now consider $f \simeq g_1(\iota_*)\colon A \bowtie A \to *$. As said before, in this case we have $\operatorname{hcat}(f) = \operatorname{relcat}_1(\iota_X)$. So we get $\operatorname{secat}(f) = 0 < \operatorname{relcat}(\iota_*) = 1 < \operatorname{hcat}(f) = \operatorname{relcat}_1(\iota_*) = 2$.

2. Hopf invariant and homotopy pushout

Let us consider any homotopy commutative square:

$$(\ddagger)\qquad \begin{tikzcd}
\Sigma_A W \arrow[r, "\rho"] \arrow[d, "f"'] & B \arrow[d, "\kappa_Y"] \\
X \arrow[r, "\chi"'] & Y
\end{tikzcd}$$

Proposition 8. *The homotopy commutative square above can be splitted into the following homotopy commutative diagram:*

$$\begin{tikzcd}
\Sigma_A W \arrow[r] \arrow[d, "\rho"'] \arrow[rrr, bend left, "f"] & G_1(\iota_X) \arrow[r] \arrow[d] & G_n(\iota_X) \arrow[r] \arrow[d] & X \arrow[d, "\chi"] \\
B \arrow[r] \arrow[rrr, bend right, "\kappa_Y"'] & G_1(\kappa_Y) \arrow[r] & G_n(\kappa_Y) \arrow[r] & Y
\end{tikzcd}$$

Proof. Set $\phi = \rho \circ \theta$. Since $\theta \circ \eta \simeq \theta \circ \beta$, also $\phi \circ \eta \simeq \phi \circ \beta$. First notice that we can insert the original homotopy square inside the following homotopy commutative diagram:

$$\begin{tikzcd}
 & W \arrow[rr, "\eta"] \arrow[dl, "\beta"'] \arrow[dd] & & A \arrow[dl] \arrow[dr, "\iota_X"] \arrow[dd, "\phi"] & \\
A \arrow[rr] \arrow[dd, "\phi"'] & & \Sigma_A W \arrow[rr, "f"'] \arrow[dd, "\rho"] & & X \arrow[dd, "\chi"] \\
 & B \arrow[rr, equal] \arrow[dl, equal] & & B \arrow[dl, equal] \arrow[dr] & \\
B \arrow[rr, equal] & & B \arrow[rr, "\kappa_Y"'] & & Y
\end{tikzcd}$$

By induction on $n \geqslant 1$, starting from the outside cube of the above diagram and $\phi_0 = \phi$, we can build a homotopy diagram:

$$\begin{tikzcd}
W \arrow[rr] \arrow[dr] \arrow[dd] & & F_{n-1}(\iota_X) \arrow[rr] \arrow[dl] \arrow[dd, dashed] & & A \arrow[dl] \arrow[dr, "\iota_X"] \arrow[dd, "\phi"] & \\
 & G_{n-1}(\iota_X) \arrow[rr] \arrow[dd, "\phi_{n-1}"] & & G_n(\iota_X) \arrow[rr] \arrow[dd, dotted, "\phi_n"] & & X \arrow[dd, "\chi"] \\
B \arrow[rr] \arrow[dr] & & F_{n-1}(\kappa_Y) \arrow[rr] \arrow[dl] & & B \arrow[dl] \arrow[dr, "\kappa_Y"] & \\
 & G_{n-1}(\kappa_Y) \arrow[rr] & & G_n(\kappa_Y) \arrow[rr, "g_n"'] & & Y
\end{tikzcd}$$

where the dashed and dotted maps are induced by the homotopy pullback $F_{n-1}(\kappa_Y)$ and the homotopy pushout $G_n(\iota_X)$ respectively.

So we obtain a homotopy commutative diagram:

$$\begin{tikzcd}
 & W \arrow[rr] \arrow[dl] \arrow[dd] & & A \arrow[dl] \arrow[dr, "\iota_X"] \arrow[dd] & \\
A \arrow[rr] \arrow[dd] & & G_n(\iota_X) \arrow[rr, "g_n"'] \arrow[dd, dotted] & & X \arrow[dd, "\chi"] \\
 & B \arrow[rr, equal] \arrow[dl, equal] & & B \arrow[dl] \arrow[dr] & \\
B \arrow[rr] & & G_n(\kappa_Y) \arrow[rr, "g_n"'] & & Y
\end{tikzcd}$$

Finally take the homotopy pushout inside the upper and lower lefter squares to get the homotopy commutative diagram:

$$\begin{tikzcd}
\Sigma_A W \arrow[r, "\omega_n"'] \arrow[d, "\rho"'] & G_n(\iota_X) \arrow[r] \arrow[d] & X \arrow[d, "\chi"] \\
B \arrow[r] & G_n(\kappa_Y) \arrow[r] & Y
\end{tikzcd}$$

and this gives the required splitting of the original square. $\square$

Proposition 9. *If the square ‡ is a homotopy pushout, then*

$$\operatorname{relcat}(\kappa_Y) \leqslant \operatorname{hcat}(f).$$

As a particular case, when $B \simeq *$, Y is the homotopy cofibre of f, and $\operatorname{relcat}(\kappa_Y) = \operatorname{cat}(Y)$. So the Proposition asserts that $\operatorname{hcat}(f) \geqslant \operatorname{cat}(Y)$.

Proof. Let $\mathrm{hcat}(f) \leqslant n$, so we have a homotopy section σ of $g_n(\iota_X)$ such that $\sigma \circ f \simeq \omega_n$. First apply the 'Whisker maps inside a cube' lemma to the outer part of the following homotopy commutative diagram:

$$\begin{tikzcd}
& \Sigma_A W \arrow[rr] \arrow[dl, "\omega_n"'] \arrow[dd, equal] & & B \arrow[dl, "a"'] \arrow[dd, equal] \arrow[dr, "\alpha_n"] & \\
G_n(\iota_X) \arrow[rr] \arrow[dd] & & S \arrow[rr, dotted, "b"'] \arrow[dd, dotted, "c"'] & & G_n(\kappa_Y) \arrow[dd, "g_n"] \\
& \Sigma_A W \arrow[rr] \arrow[dl] & & B \arrow[dl] \arrow[dr] & \\
X \arrow[rr] & & Y \arrow[rr, equal] & & Y
\end{tikzcd}$$

where the inner horizontal squares are homotopy pushouts, and c and b are the whisker maps induced by the homotopy pushout S. Next build the following homotopy commutative diagram:

$$\begin{tikzcd}
& \Sigma_A W \arrow[rr] \arrow[dl, "f"'] \arrow[dd, equal] & & B \arrow[dl, "\kappa_Y"'] \arrow[dd, equal] & \\
X \arrow[rr] \arrow[dd, "\sigma"'] & & Y \arrow[dd, dotted, "d"'] & & \\
& \Sigma_A W \arrow[rr] \arrow[dl, "\omega_n"'] & & B \arrow[dl, "a"] \arrow[dr, "\alpha_n"] & \\
G_n(\iota_X) \arrow[rr] & & S \arrow[rr, "b"'] & & G_n(\kappa_Y)
\end{tikzcd}$$

where d is the whisker map induced by the homotopy pushout Y. Let $\sigma' = b \circ d$. We have $g_n \circ \sigma' \simeq g_n \circ b \circ d \simeq c \circ d \simeq \mathrm{id}_C$ and $\sigma' \circ \kappa_Y \simeq b \circ d \circ \kappa_Y \simeq b \circ a \simeq \alpha_n$. $\square$

Corollary 10. *In the diagram* ‡*, if* $\mathrm{relcat}\,(\kappa_Y) = \mathrm{relcat}\,(\iota_X) + 1$*, then* $\mathrm{hcat}(f) = \mathrm{relcat}\,(\iota_X) + 1$*.*

Proof. By Proposition 9, the hypothesis implies that $\mathrm{hcat}(f) \geqslant \mathrm{relcat}\,(\iota_X) + 1$. But by Proposition 6, we have $\mathrm{hcat}(f) \leqslant \mathrm{relcat}\,(\iota_X) + 1$. So we have the equality. $\square$

It is now easy to exhibit examples of maps f with $\mathrm{hcat}(f) = \mathrm{relcat}\,(\iota_X) + 1$. Indeed there are plenty examples of homotopy pushouts where $\mathrm{relcat}\,(\kappa_Y) = \mathrm{relcat}\,(\iota_X) + 1$:

Example 11. Let $A = B = *$ and $f\colon S^r \to S^n$ be any of the Hopf maps $S^3 \to S^2$, $S^7 \to S^4$ or $S^{15} \to S^8$. So here $\mathrm{relcat}\,(\iota_X) = \mathrm{cat}\,(S^n) = 1$. On the other hand it is well known that those maps have a homotopy cofibre S^n/S^r of category 2, so here $\mathrm{relcat}\,(\kappa_Y) = \mathrm{cat}\,(S^n/S^r) = 2$. By Corollary 10, we have $\mathrm{hcat}(f) = 2$.

Example 12. Let f be the map u in the homotopy cofibration

$$Z \bowtie Z \xrightarrow[u]{} \Sigma Z \vee \Sigma Z \xrightarrow[t_1]{} \Sigma Z \times \Sigma Z$$

where $Z \bowtie Z \simeq \Sigma(Z \wedge Z)$ is the join of two copies of Z and is also the suspension of the smash product of two copies of Z. Let $A = B = *$, $\Sigma Z \not\simeq *$. We have $\mathrm{relcat}\,(\iota_X) = \mathrm{cat}\,(\Sigma Z \vee \Sigma Z) = 1$ and $\mathrm{relcat}\,(\kappa_Y) = \mathrm{cat}\,(\Sigma Z \times \Sigma Z) = 2$, so by Corollary 10 again, we have $\mathrm{hcat}(u) = 2$.

Example 13. For $i \geqslant 1$, let f be the map β_i in the Ganea construction:

$$\begin{tikzcd}
A \arrow[r, "\theta_i"] \arrow[dr, "\alpha_i"'] & F_i \arrow[r, "\eta_i"] \arrow[d, "\beta_i"] & A \arrow[d, "\alpha_{i+1}"] \\
& G_i \arrow[r, "\gamma_i"'] & G_{i+1}
\end{tikzcd}$$

Actually F_i is a join over A of $i+1$ copies of F_0, and also a relative suspension $\Sigma_A W$ where W is a relative smash product. For any $i \leqslant \operatorname{relcat}(\iota_X)$, we have $\operatorname{relcat}(\alpha_i) = i$, see [2], Proposition 23. So by Corollary 10 again, if $i < \operatorname{relcat}(\iota_X)$, we have $\operatorname{hcat}(\beta_i) = \operatorname{relcat}(\alpha_i) + 1 = i + 1$.

3. The Strong Hopf category

In [2], we introduced the strong version of relcat , namely Relcat . In this section, we introduce the strong version of hcat, namely Hcat. This gives an alternative way, sometimes usefull, to see if a map has a Hopf category less or equal to n. Also this will lead to a new inequality: $\operatorname{hcat}(f) \geqslant \operatorname{relcat}(f)$. Consequently, if $\operatorname{relcat}(f) > \operatorname{relcat}(\iota_X)$, then $\operatorname{hcat}(f) = \operatorname{relcat}(\iota_X) + 1$.

Definition 14. The *strong Hopf category of a map* $f\colon \Sigma_A W \to X$ is the least integer $n \geqslant 1$ such that:

- there are maps $\iota_0\colon A \to X_0$ and a homotopy inverse $\lambda\colon X_0 \to A$, i.e. $\iota_0 \circ \lambda \simeq \mathrm{id}_{X_0}$ and $\lambda \circ \iota_0 \simeq \mathrm{id}_A$;
- for each i, $0 \leqslant i < n$, there is a homotopy commutative cube:

$$(\natural)\qquad\begin{tikzcd}
& W \arrow[rr, "\beta"] \arrow[dl, "\eta"'] \arrow[dd] & & A \arrow[dl] \arrow[dd, "\iota_i"] \\
A \arrow[rr] \arrow[dd, equal] & & \Sigma_A W \arrow[dd, "\zeta_{i+1}"] & \\
& Z_i \arrow[rr, "z_i"] \arrow[dl] & & X_i \arrow[dl, "\chi_i"] \\
A \arrow[rr, "\iota_{i+1}"'] & & X_{i+1} &
\end{tikzcd}$$

  where the bottom square is a homotopy pushout.
- $X_n = X$ and $\zeta_n \simeq f$.

We denote this integer by $\operatorname{Hcat}(f)$.

Notice that $\iota_{i+1} \simeq \zeta_{i+1} \circ \theta \simeq \chi_i \circ \iota_i$. In particular, this means that $\operatorname{Pushcat}(\iota_i) \leqslant i$ in the sense of [2], Definition 3.

For $0 \leqslant i \leqslant n$, define the sequence of maps $\xi_i\colon X_i \to X$ with the relation $\xi_i = \xi_{i+1} \circ \chi_i$ (when $i < n$), starting with $\xi_n = \mathrm{id}_X$. We have $\xi_n \circ \iota_n \simeq \iota_X$ and $\xi_i \circ \iota_i = \xi_{i+1} \circ \chi_i \circ \iota_i \simeq \xi_{i+1} \circ \iota_{i+1} \simeq \iota_X$ by decreasing induction. Also $\iota_X \circ \lambda \simeq \xi_0 \circ \iota_0 \circ \lambda \simeq \xi_0$. Moreover, for $0 < i \leqslant n$ we have we have $\xi_i \circ \zeta_i \simeq f$ by the

'Whisker maps inside a cube lemma'. So we have the following homotopy diagram:

$$\begin{tikzcd}
& W \arrow[rr, "\eta"] \arrow[dl, "\beta"'] \arrow[dd] & & A \arrow[dl] \arrow[dr, "\theta"] \arrow[dd, equal] & \\
A \arrow[rr] \arrow[dd, "\iota_i"'] & & \Sigma_A W \arrow[rr, equal] \arrow[dd, "\zeta_{i+1}"] & & \Sigma_A W \arrow[dd, "f"] \\
& Z_i \arrow[rr] \arrow[dl] & & A \arrow[dl, "\iota_{i+1}"] \arrow[dr, "\iota_X"] & \\
X_i \arrow[rr, "\chi_i"'] & & X_{i+1} \arrow[rr, "\xi_{i+1}"'] & & X
\end{tikzcd}$$

We say that a map $g\colon B \to Y$ is 'relatively dominated' by a map $f\colon B \to X$ if there is a map $\varphi\colon X \to Y$ with a homotopy section $\sigma\colon Y \to X$ such that $\varphi \circ f \simeq g$ and $\sigma \circ g \simeq f$.

Proposition 15. *A map* $g\colon \Sigma_A W \to Y$ *has* $\mathrm{hcat}(g) \leqslant n$ *iff* g *est relatively dominated by a map* $f\colon \Sigma_A W \to X$ *with* $\mathrm{Hcat}(f) \leqslant n$.

Proof. Consider the map $\omega_n\colon \Sigma_A W \to G_n(\iota_Y)$ as in diagram † and notice that $\mathrm{Hcat}(\omega_n) \leqslant n$. If $\mathrm{hcat}(f) \leqslant n$, then f is relatively dominated by ω_n.

For the reverse direction, by hypothesis, we have a map φ and a homotopy section σ such that $\varphi \circ f \simeq g$ and $\sigma \circ g \simeq f$; composing with θ, we have also $\varphi \circ \iota_X \simeq \iota_Y$ and $\sigma \circ \iota_Y \simeq \iota_X$. From the hypothesis $\mathrm{Hcat}(f) \leqslant n$, we get a sequence of homotopy commutative diagrams, for $0 \leqslant i < n$, which gives the top part of the following diagram.

We show by induction that the map $\varphi \circ \xi_i\colon X_i \to Y$ factors through $g_i\colon G_i(\iota_Y) \to Y$ up to homotopy. This is true for $i = 0$ since we have $\xi_0 \simeq \iota_X \circ \lambda$, so $\varphi \circ \xi_0 \simeq \varphi \circ \iota_X \circ \lambda \simeq \iota_Y \circ \lambda = g_0 \circ \lambda$. Suppose now that we have a map $\lambda_i\colon X_i \to G_i(\iota_Y)$ such that $g_i \circ \lambda_i \simeq \varphi \circ \xi_i$. Then we construct a homotopy commutative diagram

$$\begin{tikzcd}
& Z_i \arrow[rr] \arrow[dl, "z_i"'] \arrow[dd, dashed] & & A \arrow[dl, "\iota_{i+1}"'] \arrow[dr] \arrow[dd, equal] & \\
X_i \arrow[rr] \arrow[dd, "\lambda_i"'] & & X_{i+1} \arrow[rr, "\xi_{i+1}"'] \arrow[dd, dotted, "\lambda_{i+1}"] & & X \arrow[dd, "\varphi"] \\
& F_i \arrow[rr] \arrow[dl] & & A \arrow[dl, "\alpha_{i+1}"] \arrow[dr] & \\
G_i(\iota_Y) \arrow[rr] & & G_{i+1}(\iota_Y) \arrow[rr, "g_{i+1}"'] & & Y
\end{tikzcd}$$

where $Z_i \dashrightarrow F_i$ is the whisker map induced by the bottom homotopy pullback and $\lambda_{i+1}\colon X_{i+1} \cdots\!\!> G_{i+1}(\iota_Y)$ is the whisker map induced by the top homotopy pushout. The composite $g_{i+1} \circ \lambda_{i+1}$ is homotopic to $\varphi \circ \xi_{i+1}$. Hence the inductive step is proven.

At the end of the induction, we have $g_n \circ \lambda_n \simeq \varphi \circ \xi_n = \varphi \circ \mathrm{id}_X = \varphi$. As we have a homotopy section $\sigma\colon Y \to X_n = X$ of φ, we get a homotopy section $\lambda_n \circ \sigma$ of g_n. Moreover, we have $(\lambda_n \circ \sigma) \circ g \simeq \lambda_n \circ f \simeq \lambda_n \circ \zeta_n \simeq \omega_n$. □

Example 16. If we consider any relative suspension $\Sigma_A f\colon \Sigma_A W \to \Sigma_A Z$ (and in particular, of course, when $A = *$, any suspension $\Sigma f\colon \Sigma W \to \Sigma Z$), we have $\mathrm{Hcat}(\Sigma_A f) = 1$. And so, any map g that is relatively dominated by a (relative) suspension has $\mathrm{hcat}(g) = 1$.

In fact, by definition, a map g has $\mathrm{Hcat}(g) = 1$ if and only if g is a (relative) suspension. There are maps for which the strong Hopf category is greater than the Hopf category: For instance, consider the null map $f\colon * \to X$ of Example 5; if X is a space with $\mathrm{cat}\,(X) = 1$ that is not a suspension, then f cannot be a suspension, so $\mathrm{Hcat}(f) > \mathrm{hcat}(f) = 1$.

Proposition 17. *In the diagram* $\natural$*, we have*

$$\mathrm{Relcat}\,(\zeta_i) \leqslant i$$

As ω_i is a particular case of ζ_i, this implies $\mathrm{Relcat}\,(\omega_i) \leqslant i$.

Proof. For $i > 0$, let build the following homotopy diagram where the three squares are homotopy pushouts:

$$\begin{tikzcd}
W \arrow[r] \arrow[d, "\beta"'] \arrow[rr, bend left, "\eta"] & Z_{i-1} \arrow[r] \arrow[d] \arrow[dd, bend left, "z_{i-1}"] & A \arrow[d, "\theta"'] \arrow[dd, bend left, "\iota_i"] \\
A \arrow[r] \arrow[dr, bend right, "\iota_{i-1}"'] & C_{i-1} \arrow[r] \arrow[d, "c_{i-1}"'] & \Sigma_A W \arrow[d, "\zeta_i"'] \\
 & X_{i-1} \arrow[r] & X_i
\end{tikzcd}$$

and where the map $c_{i-1} = (\iota_{i-1}, z_{i-1})$ is the whisker map induced by the homotopy pushout.

We have $\mathrm{secat}\,(\iota_{i-1}) \leqslant \mathrm{Pushcat}\,(\iota_{i-1}) \leqslant i-1$ by [2], Theorem 18. So $\mathrm{secat}\,(c_{i-1}) \leqslant i-1$ by [2], Proposition 29. So $\mathrm{Relcat}\,(c_{i-1}) \leqslant (i-1)+1 = i$ by [2], Theorem 18. And this implies $\mathrm{Relcat}\,(\zeta_i) \leqslant i$ by [2], Lemma 11. □

Theorem 18. *For any* $f\colon \Sigma_A W \to X$*, we have*

$$\mathrm{Relcat}\,(f) \leqslant \mathrm{Hcat}(f) \quad \textit{and} \quad \mathrm{relcat}\,(f) \leqslant \mathrm{hcat}(f)$$

Proof. If $\mathrm{Hcat}(f) = n$, then we have $f \simeq \zeta_n$ in $\natural$. So $\mathrm{Relcat}\,(f) = \mathrm{Relcat}\,(\zeta_n) \leqslant n$ by Proposition 17.

If $\mathrm{hcat}(f) = n$, then f is relatively dominated by ω_n. As $\mathrm{Relcat}\,(\omega_n) \leqslant n$, we have $\mathrm{relcat}\,(f) \leqslant n$ by [2], Proposition 10. □

As a corollary, we get an indirect proof of Proposition 9 because $\mathrm{relcat}\,(\kappa_Y) \leqslant \mathrm{relcat}\,(f)$ by [2], Lemma 11, that asserts that a homotopy pushout doesn't increase the relative category.

It is not difficult to find an example where these inequalities are strict:

Example 19. Let f be the map t_1 in the homotopy cofibration

$$Z \bowtie Z \xrightarrow[u]{} \Sigma Z \vee \Sigma Z \xrightarrow[t_1]{} \Sigma Z \times \Sigma Z$$

Let $A = *$, $\Sigma Z \not\simeq *$. As t_1 is a homotopy cofibre, we have $\mathrm{relcat}\,(t_1) \leqslant \mathrm{Relcat}\,(t_1) \leqslant 1$, see [2], Proposition 9. On the other hand, we have $\mathrm{Hcat}(t_1) \geqslant \mathrm{hcat}(t_1) \geqslant \mathrm{relcat}\,(\iota_X) = \mathrm{cat}\,(\Sigma Z \times \Sigma Z) = 2$ by Proposition 6.

4. Equivalent conditions to get the Hopf category

Let be given any map $f\colon \Sigma_A W \to X$ with $\mathrm{secat}\,(\iota_X) \leqslant n$ and any homotopy section $\sigma\colon X \to G_n$ of $g_n\colon G_n \to X$. Consider the following homotopy pullbacks:

$$\begin{tikzcd}
Q \arrow[r, "\pi"] \arrow[d, "\pi'"'] & \Sigma_A W \arrow[d, "\theta_n^W"'] \arrow[dr, equal] & \\
\Sigma_A W \arrow[r, "\bar\sigma"'] \arrow[d, "f"'] & H_n \arrow[r, "\eta_n^W"'] \arrow[d, "f_n"'] & \Sigma_A W \arrow[d, "f"] \\
X \arrow[r, "\sigma"'] & G_n \arrow[r, "g_n"'] & X
\end{tikzcd}$$

where $\theta_n^W = (\omega_n, \mathrm{id}_{\Sigma_A X})$ is the whisker map induced by the homotopy pullback H_n. By the 'Prism lemma' (see [2], Lemma 46, for instance), we know that the homotopy pullback of σ and f_n is indeed $\Sigma_A W$, and that $\eta_n^W \circ \bar\sigma \simeq \mathrm{id}_{\Sigma_A W}$. Also notice that $\pi \simeq \pi'$ since $\pi \simeq \eta_n^W \circ \theta_n^W \circ \pi \simeq \eta_n^W \circ \bar\sigma \circ \pi' \simeq \pi'$.

Proposition 20. *Let be given any map $f\colon \Sigma_A W \to X$ with* $\mathrm{secat}\,(\iota_X) \leqslant n$ *and any homotopy section $\sigma\colon X \to G_n(\iota_X)$ of $g_n\colon G_n(\iota_X) \to X$. With the same definitions and notations as above, the following conditions are equivalent:*

(i) $\sigma \circ f \simeq \omega_n$.
(ii) π *has a homotopy section.*
(iii) π *is a homotopy epimorphism.*
(iv) $\theta_n^W \simeq \bar\sigma$.

Proof. We have the following sequence of implications:

(i) $\Longrightarrow$ (ii): Since $\sigma \circ f \simeq \omega_n \simeq f_n \circ \theta_n^W \circ \mathrm{id}_{\Sigma_A W}$, we have a whisker map $(f, \mathrm{id}_{\Sigma_A W})\colon \Sigma_A W \to Q$ induced by the homotopy pullback Q which is a homotopy section of π.

(ii) $\Longrightarrow$ (iii): Obvious.

(iii) $\Longrightarrow$ (iv): We have $\theta_n^W \circ \pi \simeq \bar\sigma \circ \pi$ since $\pi \simeq \pi'$. Thus $\theta_n^W \simeq \bar\sigma$ since π is a homotopy epimorphism.

(iv) $\Longrightarrow$ (i): We have $\sigma \circ f \simeq f_n \circ \bar\sigma \simeq f_n \circ \theta_n^W \simeq \omega_n$. □

Theorem 21. *Let be a $(q-1)$-connected map $\iota_X\colon A \to X$ with* $\mathrm{secat}\,\iota_X \leqslant n$. *If $\Sigma_A W$ is a CW-complex with* $\dim \Sigma_A W < (n+1)q - 1$ *then $\sigma \circ f \simeq \omega_n$ for any homotopy section σ of g_n.*

Proof. Recall that g_i is the $(i+1)$-fold join of ι_X. Thus by [4], Theorem 47, we obtain that, for each $i \geqslant 0$, $g_i : G_i \to X$ is $(i+1)q-1$-connected. As g_i and η_i^W have the same homotopy fibre, the Five lemma implies that $\eta_i^W : H_i \to \Sigma_A W$ is $(i+1)q-1$-connected, too. By [5], Theorem IV.7.16, this means that for every CW-complex K with $\dim K < (i+1)q-1$, η_i^W induces a one-to-one correspondence $[K, H_i] \to [K, \Sigma_A W]$. Apply this to $K = \Sigma_A W$ and $i = n$: Since θ_n^W and $\bar\sigma$ are both homotopy sections of η_n^W, we obtain $\theta_n^W \simeq \bar\sigma$, and Proposition 20 implies the desired result. □

Example 22. Let $A = *$ and $W = S^{r-1}$, so $\Sigma_A W = S^r$, and $X = S^m$. In this case $\mathrm{secat}\,\iota_X = \mathrm{cat}\, S^m = 1$. Hence Theorem 21 means that if $r < 2m - 1$, we have $\sigma \circ f \simeq \omega_1$, whatever can be f and $\sigma\colon X \to G_1(\iota_X)$, so $\mathrm{hcat} f = 1$ and we get

by Proposition 9 that the homotopy cofibre C of f has $\operatorname{cat} C \leqslant 1$. (Notice that if $r < m$ then f is a nullhomotopic, so C is simply $S^m \vee S^{r+1}$.)

Example 23. Let $A = *$, $\Sigma W \simeq \Sigma(S^{r-1} \vee S^{r-1}) \simeq S^r \vee S^r$, $X \simeq S^r \times S^r$ and consider $t_1 \colon S^r \vee S^r \to S^r \times S^r$. Here $\operatorname{secat}(\iota_X) = \operatorname{cat}(S^r \times S^r) = 2$. For any $r \geqslant 1$, we have $\dim(S^r \vee S^r) = r < (2+1)r - 1$, so $\operatorname{hcat}(t_1) = 2$.

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